ON THE MODULI OF KÄHLER-EINSTEIN FANO MANIFOLDS

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Abstract. The existence of moduli algebraic space of Kähler-Einstein Fano manifolds with finite automorphism groups follows from a conjecture of Tian on the α-invariant, under certain hypotheses.

1. Introduction

Let us begin with the following background.

Conjecture 1.1 (Yau-Tian-Donaldson conjecture)**.** *A Fano manifold X has Kähler-Einstein metrics if and only if $(X, \mathcal{O}_X(-K_X))$ is K-polystable.*

The K-polystability notion was introduced by [Tia97], reformulated and generalized by [Don02]. The "only if" direction is proved by [Mab08], [Mab09], [Ber12] for fully general case. On the converse direction, it seems that after [Don11] which proposed a new continuity method using conical Kähler-Einstein metrics, there are many developments (e.g. [Ber10], [Bre11], [JMR11], [Sun11] etc) and an affirmative solution is announced in [CDS12].

On the other hand, the existence of Kähler-Einstein metric or K-polystability for a given Fano manifold and the behaviour with respect to complex deformation have both certain independent aspects from Conjecture 1.1 itself and they are highly non-trivial questions in general. The purpose of this paper is to discuss the latter question and consider moduli spaces of Kähler-Einstein Fano manifolds.

Roughly speaking, we prove that for Fano manifolds with discrete automorphism groups, the existence of Kähler-Einstein metric is Zariski open condition and those Fano manifolds form separated moduli algebraic spaces in the sense of [Art71], both under certain hypotheses.

As a preparation, let us recall the definition of the algebro-geometric counterpart of the α-invariant ([Tia87]) i.e., the so-called *global log-canonical threshold* ([Sho92]) and its variants as follows.

Definition 1.2. Suppose X is a projective manifold, E is an effective $\mathbb{Q}$-divisor on X, and L is an ample line bundle on X. Then we consider the following invariants.

$$\mathrm{lct}(X,E) := \sup\{\alpha > 0 \mid (X,\alpha E) \text{ is log canonical}\},$$

$$\mathrm{lct}_l(X;L) := \inf_{D\in|lL|}\mathrm{lct}\big(X,\frac{1}{l}D\big),$$

and

$$\mathrm{glct}(X;L) := \inf_{l\in\mathbb{Z}_{>0}}\mathrm{lct}_l(X;L),$$

which are called the *log canonical threshold* of X with respect to E, the *log canonical threshold* of X with respect to L with exponent m, and the *global log canonical threshold* of X with respect to L.

For a Fano manifold X, we simply write $\mathrm{glct}(X) := \mathrm{glct}(X;-K_X)$, $\mathrm{lct}_l(X) := \mathrm{lct}_l(X;-K_X)$.

Note that the last notion corresponds to α-invariant (cf. [Tia87], [CSD08]). Recall that there are following fundamental conjectures by Tian [Tia12, Conjecture 5.3, 5.4].

Conjecture 1.3 ([Tia12, Conjecture 5.3])**.** *For an arbitrary polarized projective manifold* (X,L)*,* $\mathrm{lct}(X;L) = \mathrm{lct}_l(X;L)$ *for* $l \gg 0$*.*

Conjecture 1.4 ([Tia12, Conjecture 5.4])**.** *For an arbitrary polarized projective manifold* (X,L)*,* $\mathrm{lct}(X) = \mathrm{lct}_l(X)$ *for* $l \geq l_0$ *where* $\oplus_{0\leq i\leq l_0}H^0(X,L^{\otimes i})$ *generates the homogeneous coordinate ring* $\oplus_{i\geq 0}H^0(X,L^{\otimes i})$*.*

We need a following version, weaker than Conjecture 1.4 but stronger than Conjecture 1.3.

Conjecture 1.5. *For an arbitrary flat projective family of polarized manifolds* $\pi\colon(\mathcal{X},\mathcal{L})\to S$ *with finite type algebraic scheme* S*, there is* $l\in\mathbb{Z}_{>0}$ *such that*

$$\mathrm{lct}(\mathcal{X}_s,\mathcal{L}_s) = \mathrm{lct}_l(\mathcal{X}_s,\mathcal{L}_s)$$

for any closed point $s\in S$*.*

It is obvious that Conjecture 1.5 implies Conjecture 1.3 and on the other hand, in the context of Conjecture 1.5, the $\mathcal{O}_S$-algebra $\oplus_{i\geq 0}(\pi_*\mathcal{L}^{\otimes i})$ is generated by $\oplus_{0\leq i\leq l}(\pi_*\mathcal{L}^{\otimes i})$ with some $l\in\mathbb{Z}_{>0}$ so that we can apply Conjecture 1.4 to prove Conjecture 1.5.

We also need the following technical hypothesis, which the author expects to be going to be proved in the proofs of Conjecture 1.1 (cf. [CDS12], [Don11, Conjecture 2]).

Hypothesis 1.6. *For a Fano manifold X, if X does not admit Kähler-Einstein metric, the following holds. For an arbitrary smooth divisor $D \in |-\mu K_X|$ with some $\mu \in \mathbb{Z}_{>0}$, we have a non-trivial normal log test configuration $(\mathcal{X}, \mathcal{D})$ of $(X, (1-\beta_0)D)$ with angle $2\pi\beta_0$ ($1 - \frac{1}{\mu} < \beta_0 \leq 1$) such that $\mathrm{DF}_{\beta_0}((\mathcal{X}, \mathcal{D}); -K_{\mathcal{X}}) = 0$ and $\mathcal{X}_0$ is $\mathbb{Q}$-Fano variety. Moreover, the $\mathbb{Q}$-Gorenstein index of $\mathcal{X}_0$ can be taken uniformly if we have a fixed dimension $\dim(X)$, μ and a lower bound for β_0 which is bigger than $1 - \frac{1}{\mu}$.*

Note that the bound of index of $\mathcal{X}_0$ is equivalent to that it can be a flat limit of (smooth) Fano manifolds inside a fixed Hilbert scheme, by the recent boundedness result [HMX12, Corollary 1.8]. Indeed, the analogical bound of $\mathbb{Q}$-Gorenstein index of the limit was indeed already proved in non-log setting ([DS12]). Now we can precisely state the main theorem.

Theorem 1.7 (Main theorem)**.** *For a flat projective family of Fano manifolds $\pi\colon \mathcal{X} \to S$, suppose that the followings are true:*

- (i) *Conjecture 1.1 holds for $\mathcal{X}_s$ for all $s \in S$,*
- (ii) *Hypothesis 1.6 holds for $\mathcal{X}_s$ for all $s \in S$,*
- (iii) *Conjecture 1.5 holds for $(\mathcal{X}, -K_{\mathcal{X}})$ and $(\mathcal{D}, -K_{\mathcal{X}}|_{\mathcal{D}})$ for some relatively smooth pluri-anticanonical Cartier divisor, i.e., an effective Cartier divisor $\mathcal{D}$ on $\mathcal{X}$ which is flat and smooth over S such that $\mathcal{D}|_s \cong -\mu K_{\mathcal{X}_s}$ with some $\mu \in \mathbb{Z}_{>0}$.*

Then the subset:

$$\{s \in S \mid \mathcal{X}_s \text{ is a KE Fano manifold with } \#\,\mathrm{Aut}(X) < \infty\} \subset S$$

is a Zariski open subset.

Note that this Zariski openness question is discussed in Donaldson's note [Don09] which took a different approach. Indeed, the recent discussion with Donaldson on this question is also reflected in this paper as we will explain.

The main corollary is the following existence of moduli space.

Corollary 1.8. *Suppose that Conjecture 1.1, 1.5 and the hypothesis 1.6 holds for some flat family (over not necessarily connected scheme) which includes all Fano n-folds and those members. Then the moduli functor of Fano manifolds with Kähler-Einstein metrics and finite automorphism groups have separated coarse moduli algebraic space.*

2. Proofs

Proof of Theorem 1.7. To state the proof, we need to recall the following characterization of test configurations and a certain modification of the K-stability notion.

Proposition 2.1 ([RT07, Proposition 3.7])**.** *Let X be a Fano manifold. Then for a large enough $m \in \mathbb{Z}_{>0}$, a one-parameter subgroup of* $\mathrm{GL}(H^0(X, \mathcal{O}_X(-mK_X)))$ *is equivalent to the data of a test configuration $(\mathcal{X}, \mathcal{L})$ of $(X, -K_X)$ whose polarization $\mathcal{L}$ is very ample (over $\mathbb{A}^1$) with exponent m.*

Definition 2.2. A $\mathbb{Q}$-Fano variety X is said to be K_m-stable (resp., K_m-semistable) if and only if $\mathrm{DF}(\mathcal{X}, \mathcal{L}) > 0$ for any test configuration of the above type (i.e., with very ample $\mathcal{L}$ of exponent m) which is non-trivial test configuration whose central fiber is normal.

Notice that [LX11] proved can be rephrased as that a $\mathbb{Q}$-Fano variety is K-stable (resp., K-polystable, K-semistable) if and only if it is K_m-stable (resp., K_m-polystable, K_m-semistable) for sufficiently divisible $m \in \mathbb{Z}_{>0}$. The name "K_m-stability" followed a suggestion of Donaldson in a discussion. The equivalent notion was also treated in [Tia12]. As we assumed Conjecture 1.5 for our $\pi\colon \mathcal{X} \to S$, we have a uniform lower bound for $\mathrm{lct}(\mathcal{X}_s; -K_{\mathcal{X}_s})$ and $\mathrm{lct}(\mathcal{D}_s; -K_{\mathcal{X}_s})$. Indeed, for each fixed l, we have uniform lower bounds for $\mathrm{lct}_l(\mathcal{X}_s; -K_{\mathcal{X}_s})$ and $\mathrm{lct}_l(\mathcal{D}_s; -K_{\mathcal{X}_s})$ for all $s \in S$. Its proof goes as follows.

Take sufficiently divisible m and Consider the projective bundle $\mathbb{P}(\pi_*\mathcal{O}_{\mathcal{X}}(\frac{m}{\mu}\mathcal{D}))$ over S. We can regard $\mathrm{lct}(\mathcal{X}_s; E)$, where E runs through $|-mK_{\mathcal{X}_s}|$, as a function on that projective bundle. It is known [Mus02] that this function is lower semicontinuous and also only takes discrete values (cf. [HMX12, section 11]). These show the existence of uniform lower bounds for $\mathrm{lct}_l(\mathcal{X}_s; -K_{\mathcal{X}_s})$ and $\mathrm{lct}_l(\mathcal{D}_s; -K_{\mathcal{X}_s})$. Actually these uniform lower bounds are only consequences of Conjecture 1.5 which we need for the proof of Theorem 1.7.

On the other hand, let us recall the following results from [OS11].

Theorem 2.3 ([OS11, Theorem 4.1])**.** *Let X be a projective variety, D' is an effective $\mathbb{Q}$ divisor and* $0 \le \beta \le 1$. (i) *Assume $(X, (1-\beta)D)$ is a log Calabi-Yau pair, i.e., $K_X + (1-\beta)D$ is numerically equivalent to the zero divisor and it is a semi-log-canonical pair (resp. kawamata-log-terminal pair). Then, $((X, D), L)$ is logarithmically K-semistable (resp. logarithmically K-stable) with respect to the boundary's parameter β.*

(ii) *Assume $(X, (1-\beta)D')$ is a semi-log-canonical model, i.e., $K_X + (1-\beta)D'$ is ample and it is a semi-log-canonical pair. Then,*

$((X,D), K_X+(1-\beta)D')$ and $\beta\in\mathbb{Q}_{>0}$ is log K-stable with respect to the boundary's parameter β.

Theorem 2.4 (cf. [OS11, Corollary 5.5])**.** *Let X be an arbitrary $\mathbb{Q}$-Fano variety and D is an integral pluri-anti-canonical Cartier divisor $D\in|-\mu K_X|$ with some $\mu\in\mathbb{Z}_{>0}$. Then $(X,-K_X)$ is logarithmically K-stable (resp. logarithmically K-semistable) for cone angle $2\pi\beta$ with $\frac{\mu-1}{\mu}<\beta<\big(\mu-1+(\frac{n+1}{n})\min\{\mathrm{glct}(X;-K_X),\mathrm{glct}(D;-K_X|_D)\}\big)/\mu$ (resp. $\frac{\mu-1}{\mu}\leq\beta\leq\big(\mu-1+(\frac{n+1}{n})\min\{\mathrm{glct}(X;-K_X),\mathrm{glct}(D;-K_X|_D)\}\big)/\mu$).*

We make some remarks. Recall that Theorem 2.3 (i) extends and algebraically recovers [Sun11, Theorem 1.1], [Bre11, Theorem1.1] and [JMR11, Theorem 2]. In other words, at least for the case where X is smooth, D is a smooth integral divisor, the above is expected to follow from their results as well. Regarding Theorem 2.4, the original version of [OS11, Corollary 5.5] treated only $\mu=1$ case, but it is recently noticed that it can be simply extended to the above form. We also remark that the case if X and D are both smooth and $\mu=1$, Theorem 2.4 also follows from [Ber10, Theorem 1.8] combined with [Ber12, Theorem1.1].

The above two theorems imply that

Corollary 2.5. *For an arbitrary smooth Fano manifold X and a smooth pluri-anticanonical divisor $D\in|-\mu K_X|$, $(X,-K_X)$ is logarithmically K-stable (resp. logarithmically K-semistable) for cone angle $2\pi\beta$ with $0<\beta<(\mu-1+(\frac{n+1}{n})\min\{\mathrm{glct}(X;-K_X),\mathrm{glct}(D;-K_X|_D)\})/\mu$ (resp. $0\leq\beta\leq(\mu-1+(\frac{n+1}{n})\min\{\mathrm{glct}(X;-K_X),\mathrm{glct}(D;-K_X|_D)\})/\mu$).*

Combining the above lower boundedness of log canonical thresholds and Corollary 2.5, we can take uniform $0<\beta_0$ such that every Fano manifolds $\mathcal{X}_s$ $(s\in S)$ are log K-stable for cone angle $2\pi\beta_0$ with respect to the divisor Cartier $\mathcal{D}_s$.

Thanks to the existence of the uniform β_0, in turn Hypothesis 1.6 implies that there is a large enough but fixed m such that for any $s\in S$, $\mathcal{X}_s$ is Kähler-Einstein Fano manifold if and only if it is K_m-stable. Take a Hilbert scheme H which exhausts all $\mathcal{X}_s$, embedded by $|-mK_{\mathcal{X}_s}|$. Naturally we have a CM line bundle on H which we denote by λ_{CM} and there are actions of $\mathrm{SL}(h^0(-mK_{\mathcal{X}_s}))$ on both H and λ_{CM} (linearization). Denote the locus in H which parametrizes Kähler-Einstein Fano manifold $\mathcal{X}_s$ with $s\in S$ by H_{KE}, and the locus which parametrizes normal fibers by H_{normal}. It is well known that H_{normal} is a Zariski open subset of H as a general phenomenon.

Let us fix a maximal torus T of $\mathrm{SL}(h^0(-mK_{\mathcal{X}_s}))$. Recall the following known fact for usual setting in Geometric Invariant Theory [Mum65] which is for *ample* linearized line bundles.

Lemma 2.6 (cf. [Mum65, Chapter 2, Proposition 2.14])**.** *Consider the action of a reductive algebraic group G on a polarized projective scheme (H,λ) i.e. H is a projective scheme and λ is an ample linearized line bundle. Let us denote the corresponding GIT weights function with respect to a one parameter subgroup $\varphi\colon \mathbb{G}_m \to G$ at $h \in H$ by $\mu(t,\varphi;\lambda)$. Then there are some finite linear functions $\{l_i\}_{i\in I}$ on $\mathrm{Hom}_{\mathrm{alg.grp}}(\mathbb{G}_m,T)\otimes_{\mathbb{Z}}\mathbb{R}$ with rational coefficients, indexed by a finite set I such that the weight function $\mu(h,-;\lambda)$, regarded as a function from $\mathrm{Hom}_{\mathrm{alg.grp}}(\mathbb{G}_m,T)$ to $\mathbb{Z}$, extends to a piecewise linear rational function of the form*

$$\sup\{l_j(-) \mid j \in J_h\}$$

with some $J_h \subset I$. Moreover, ψ is constructible in the sense that for any $J \in I$, $\{h \in H \mid J_h = J\}$ is constructible.

Recall that the CM line bundle [PT06] (cf. also [FS90]) is a line bundle defined on the base scheme for an arbitrary polarized family and the formal GIT weight of the CM line bundle on the Hilbert scheme is exactly the corresponding Donaldson-Futaki invariant. However, unfortunately we can not directly apply to the CM line bundle λ_{CM} which is *not* necessarily ample in general. Nevertheless, Lemma 2.6 can be still extended to the case for our H and the CM line bundle λ_{CM} on it, though $\mu(t,-;\lambda_{CM})$ is not necessarily convex. To see that, first we take a Plücker polarization λ_{Pl} on H which is known to be very ample, yielding the Plücker embedding of Hilbert scheme. Take sufficiently large $l \in \mathbb{Z}_{>0}$ such that $\lambda_{CM}\otimes\lambda_{Pl}^{\otimes l}$ is also very ample. Then the point is that $\mu(t;-;\lambda_{CM}) = \mu(t;-;\lambda_{CM}\otimes\lambda_{Pl}^{\otimes l}) - l\mu(t;-;\lambda_{Pl})$. Therefore Lemma 2.6 extends to the case with our setting that Hilbert scheme is H, $L=\lambda_{CM}$ and $G=\mathrm{SL}(h^0(-mK_{\mathcal{X}_s}))$ as follows.

Lemma 2.7. *There are some finite linear rational functions $\{l_i\}_{i\in I'}$ on $\mathrm{Hom}_{\mathrm{alg.grp}}(\mathbb{G}_m,T)\otimes_{\mathbb{Z}}\mathbb{R}$ with rational coefficients, indexed by a finite set I' such that the followings hold. For each $[X\subset\mathbb{P}]$, there is $J_X \subset I'$ such that its Donaldson-Futaki invariants with respect to one parameter subgroups of $G := \mathrm{SL}(h^0(-mK_{\mathcal{X}_s})$ regarded as a function from $\mathrm{Hom}_{\mathrm{alg.grp}}(\mathbb{G}_m,T)$ to $\mathbb{Z}$, extends to a piecewise linear rational function whose pieces are $\{l_j\}_{j\in J_X}$ with some $J_X \subset I'$. Moreover, ψ is constructible in the sense that for any $J_X \subset I'$, $\{[X\subset\mathbb{P}]\in H \mid J_X = J\}$ is constructible.*

Next, let us discuss from another viewpoint which is more birational-geometric method. This is related to what Donaldson called "splitting of orbits" in [Don09]. Considering the family over S and trivialize the locally free coherent sheaf $\pi_*\mathcal{O}(\mathcal{D})$ on a Zariski open covering $\{S_i\}_i$ we have the corresponding morphism $(\sqcup S_i \times T) \to H$, which can be regarded as a rational map $(\sqcup S_i \times (\mathbb{P}^1)^r) \dashrightarrow H$. For simplicity, re-set $S := \sqcup S_i$ from now on which does not lose generality from our assertion of the theorem. Thinking of a resolution of indeterminacy, we get some blow up of $S \times (\mathbb{P}^1)^r$ along a closed subscheme inside $S \times ((\mathbb{P}^1)^r \setminus T)$. The blow up is generically isomorphic over $S \times ((\mathbb{P}^1)^r \setminus T)$, which means the GIT weights with respect to any fixed one parameter subgroups in T stay constant for some Zariski open dense subset S' of S. Replace S by $S \setminus S'$ then we can prove construct a constructible stratification of S such that each stratum's degenerations with respect to any fixed one parameter subgroup of G fits into a flat family again. This implies that those corresponding Donaldson-Futaki invariants are constant on each strata by [PT06]. We can also prove it by using Wang's interpretation of Donaldson-Futaki invariant [Wan12].

Claim 2.8. *There is a finite set I of one parameter subgroups $\{\varphi_i\}\colon \mathbb{G}_m \to T$ $(i \in I)$ such that the following is true. There is a stratification of H_{normal} by constructible subsets $\{H_J\}_{J \subset I}$, such that for any $t \in H_J \subset H_{normal}$, $\mathcal{X}_t$ is K_m-stable (resp., K_m-semistable) if and only if $\mathrm{DF}(\varphi_j; [\mathcal{X}_t \subset \mathbb{P}]) > 0$ (resp., $\mathrm{DF}(\varphi_j; [\mathcal{X}_t \subset \mathbb{P}]) \geq 0$) for all $j \in J$ where $\mathrm{DF}(\varphi_j; [\mathcal{X}_t \subset \mathbb{P}])$ actually only depends on j and J. Here $\mathrm{DF}(\varphi_j; [\mathcal{X}_t \subset \mathbb{P}])$ means the Donaldson-Futaki invariant of the test configuration of $(\mathcal{X}_t, \mathcal{O}(1) = \mathcal{O}(-mK_{\mathcal{X}_t}))$ associated to the one parameter subgroup via Proposition 2.1.*

Summarizing up, we proved that the locus of Fano manifolds which are K_m-stable (resp., K_m-semistable, K_m-unstable, not K_m-stable) with respect to the maximal torus T's action are all constructible subsets. Then the constructibility of K_m-stable locus in H follows from C. Chevalley's lemma which states that the image of constructible set by an algebro-geometric morphism is again constructible. Indeed, we can apply it to the group action morphism $\mathrm{SL}(h^0(-mK_{\mathcal{X}_s})) \times H \to H$ and the constructible subset of H which parametrizes Fano manifolds which are not K_m-stable with respect to the action of T. Then, it shows that the subset of H which parametrizes Fano manifolds with discrete automophism groups which are not K_m-stable is again constructible.

On the other hand, the deformation theory of Kähler metrics with constant scalar curvature [LS94] shows it is open with respect to the Euclidean topology. More precisely speaking, we need to take resolution

of singularities of S to apply [LS94]. Thus the locus of Kähler-Einstein Fano manifolds with fintie automorphism groups should be a Zariski open subset. We complete the proof of the main theorem.

□

We prove Corollary 1.8 as follows.

proof of Corollary 1.8 (using Theorem 1.7). Consider the Hilbert scheme H which exhausts all m-pluri-anticanonically embedded Fano n-folds with uniform $m \in \mathbb{Z}_{>0}$. By applying the previous statement of Theorem 1.7 to a Zariski open covering of universal family over H, we obtain that the locus which parametrizes Kähler-Einstein Fano manifolds form a Zariski open locus. Therefore the natural moduli functor $\mathcal{F}\colon (Sch_{/\mathbb{C}})^{op} \to (Sets)$ whose B-valued points set is defined by

$$\{\text{flat proj. } f\colon \mathcal{Y} \to B \mid [\mathcal{Y}_b \subset \mathbb{P}(H^0(-mK_{\mathcal{Y}_b}))] \in H_{KE} \text{ for } \forall b \in B\}/\cong$$

naturally gives rise to Deligne-Mumford stack which is simply a quotient stack $[H_{KE}/\operatorname{PGL}(h^0(-mK_{\mathcal{X}_s}))]$. Then the Keel-Mori theorem [KM97, Corollary 1.3] implies that we have a coarse moduli algebraic space for it. The separatedness follows from a standard argument (cf. e.g. [CS10, Lemma 8.1]). This completes the proof of Corollary 1.8.

□

Example 2.9. For instance, the so-called *Del Pezzo threefolds* X are examples for which the assumption (iii) in Theorem 1.7 is known. Moreover, some of them admit Kähler-Einstein metrics with finite automorphism groups according to the calculation of log-canonical-thresholds ([CS08], [CSD08]). It is because, for X Conjecture 1.5 is solved (the global log canonical threshold is either $1/2$ or $1/4$, cf. [CSD08]) and H which is linearly equivalent to half of the anti-canonical divisor (i.e. $-K_X \cong 2H$) can be a smooth del Pezzo surface for which the Conjecture 1.5 is solved with respect to anticanonical polarization ([Che07]).

3. A conjecture

The author expects that K-polystable objects form projective moduli in general as written below. The inspirations are mainly from the following backgrounds.

- The infinite-dimensional moment map picture on the space of complex structures by Fujiki, Donaldson ([Fuj90], [Don97], [Don04]).
- The observations ([Od08], [Od11]) that the projective moduli of general type varieties with semi-log-canonical singularities,

constructed by using the minimal model program (cf. [Kol10]) and some other projective moduli corresponds to K-stability.

- The algebraicity of Gromov-Hausdorff limits of Kähler-Einstein Fano manifolds ([DS12]) with the Gromov compactness theorem (cf. [Stp12], [OSS12]).

As a preparation, we introduce the following terminology. Let us call a numerical equivalence class of ample line bundle, a *weak polarization* and a pair of projective variety and weak polarization a *weakly polarized variety*.

Conjecture 3.1 (K-moduli)**.** *The moduli functor of K-semistable weakly polarized varieties has proper coarse moduli algebraic space. Moreover, its closed points parametrizes K-polystable weakly polarized varieties and the components are all projective schemes, because the CM line bundle descends to ample line bundles on them.*

We refer to [OSS12] for detailed discussions on $\mathbb{Q}$-Fano vareities case and explicit studies of log del Pezzo surfaces.

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